# Full Characterization of Adaptively Strong Majority Voting in Crowdsourcing

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A commonly used technique for quality control in crowdsourcing is to task the workers with examining an item and voting on whether the item is labeled correctly. To counteract possible noise in worker responses, one solution is to keep soliciting votes from more workers until the difference between the numbers of votes for the two possible outcomes exceeds a pre-specified threshold  $\delta$ . We show a way to model such  $\delta$ -margin voting consensus aggregation process using absorbing Markov chains. We provide closed-form equations for the key properties of this voting process — namely, for the quality of the results, the expected number of votes to completion, the variance of the required number of votes, and other moments of the distribution. Using these results, we show further that one can adapt the value of the threshold  $\delta$  to achieve quality-equivalence across voting processes that employ workers of different accuracy levels. We then use this result to provide efficiency-equalizing payment rates for groups of workers characterized by different levels of response accuracy. Finally, we perform a set of simulated experiments using both fully synthetic data as well as real-life crowdsourced votes. We show that our theoretical model characterizes the outcomes of the consensus aggregation process well.

Key words: crowdsourcing, labeling aggregation, majority voting, data quality control, fair remuneration, Markov random walk

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# 1. Introduction

Businesses are increasingly turning to crowdsourcing as a means of obtaining knowledge, labour, and creativity from groups of distributed workers. Workers in crowdsourcing are rarely perfect, and the literature is abundant with papers that put forth various vote aggregation schemes to ensure quality control for the crowdsourced labels (e.g., Chilton et al. (2013), Dai et al. (2013), Mortensen et al. (2013)). In practice, though, simple aggregation schemes, such as majority voting or its variations that require a stronger majority, are widely used, mainly due to their simplicity.

A commonly encountered variation of majority voting asks for multiple votes, until the positive votes are  $\delta$  votes more than the negatives (or vice versa). We will refer to this process as  $\delta$ -margin

majority voting.<sup>1</sup> While this consensus schemes of " $\delta$ -margin voting" is salient in practice, the process tends to be perceived as heuristic and adhoc, lacking a solid theoretical foundation that describes its properties.

In this work, we show that a  $\delta$ -margin voting process can be easily modeled with well-established mathematical tools (namely, Markov chains with absorbing states). Our research contribution is simple: We provide a thorough theoretical description of key properties of  $\delta$ -margin voting, allowing decision-makers to give ex-ante answers to some of the more pervasive questions in crowdsourced experiment design: How to structure a crowdsourced voting process to achieve a given label accuracy? What is the cost of running such a process? Can the same accuracy be achieved by a pool of lower accuracy workers? We provide closed-form theoretical results that quantify the quality and cost of the results.

The organization of this paper is as follows. Section 2 introduces the Markov chain formalism for the  $\delta$ -margin voting process. Section 3 overviews existing work on consensus aggregation design and quality assurance, highlighting both the novelty and relevance of our theoretical presentations. Section 4 derives the theoretical equations for key characteristics of the  $\delta$ -margin voting process: quality of the results, expected value and variance of the number of votes required to reach consensus, and the overall distribution for time to completion. Section 5 expands these results to demonstrate that  $\delta$ -margin processes with workers of different quality can have equivalent quality of results, and we further elaborate on this equivalence to propose a quality-sensitive payment scheme that connects result quality with worker accuracy. Section 6 presents experimental validation with real data comparing the theoretical predictions against results from real processes. Finally, Section 7 discusses both the immediate and the more involved research developments that can be built upon this work.

# 2. Markov chain model of $\delta$ -margin majority voting

Consider the problem of labeling items on a crowdsourcing platform using  $\delta$ -margin voting. For every item that needs to be assigned a binary label. We solicit binary votes from workers who assign the correct label with a probability p and the incorrect label with probability 1 - p. We keep asking workers to assign labels to an item, until the absolute difference between the number of votes for the two classes exceeds a predefined consensus threshold  $\delta$ .

**Example.** Set  $\delta = 2$ . Then, the process will stop when the vote-count tuples  $\langle n_{correct}, n_{incorrect} \rangle$  reach one of the following stages:  $\langle 2, 0 \rangle$ ,  $\langle 0, 2 \rangle$ ,  $\langle 3, 1 \rangle$ ,  $\langle 1, 3 \rangle$ ,  $\langle 4, 2 \rangle$ ,  $\langle 2, 4 \rangle$ , and so on. A consensus

<sup>&</sup>lt;sup>1</sup> We use the term *margin* in its colloquial meaning – an amount beyond the minimum necessary (as opposed to its usage in statistics, where it typically connotes an idea of proportion and not difference).



Figure 1 A Markov chain diagram illustrating state transitions for  $\delta$ -margin majority voting on a single item. Node labels indicate the difference between the numbers of correct  $(n_1)$  and incorrect  $(n_0)$  votes,  $n_1 - n_0$ . Consensus is reached in absorbing states  $\delta$  and  $-\delta$ , the former resulting in a correct resolution, and the latter resulting in an incorrect resolution.

vote obtained in one of the states  $\{\langle 2, 0 \rangle, \langle 3, 1 \rangle, \langle 4, 2 \rangle, ...\}$  is correct, while a consensus vote obtained in one of the states  $\{\langle 0, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 4 \rangle, ...\}$  is incorrect.

Since the only desideratum for consensus is the difference between the two types of votes, the process can be modeled as a Markov random walk. Define the current state as the difference between the numbers of correct and incorrect votes. If the difference is  $\delta$  or  $-\delta$ , the process terminates, the former resulting in a correct consensus label, and the latter in an incorrect one. In all other states, we procure an additional vote, which will be correct with probability p, which corresponds to a transition from state i to state i + 1; or it can be an incorrect vote with probability 1 - p, which makes for a transition from state i to state i - 1.

This process has a state diagram illustrated in Figure 1 and is also known as *Gambler's Ruin* model. The model is a common introductory example for random walks and describes the probability of a gambler winning a certain amount in a game of chance, vs. the probability of losing her entire gaming budget.<sup>2</sup> Surprisingly, this model has not been used to describe the process for label aggregation in crowdsourcing, and we are not aware of anyone providing results about label aggregation quality and number of votes required, for this process in crowdsourcing. Before providing, in Section 4, the analytical results that describe in closed form several key characteristics of  $\delta$ -margin voting, we provide next, in Section 3 an overview of the related literature.

#### 3. Literature review

In this section, we provide a brief overview of related work in label aggregation in crowdsourcing (Section 3.1) and then provide pointers to related work regarding  $\delta$ -margin voting, in particular (Section 3.2). To the best of our knowledge, there does not yet appear to be work that focuses on presenting theoretical results for ex-ante cost and quality estimation for  $\delta$ -margin voting.

## 3.1. Literature on quality control in label aggregation

A sizable proportion of literature on crowdsourcing proposes various quality controls for aggregating worker expertise. The majority of such work, however, is experimental (Kazai et al. 2011, Hansen et al. 2013, Yin et al. 2014), while the proposed quality control mechanisms are often ad-hoc. Although informed by empirical intuition, such quality control interventions lack theoretical guarantees (Kucherbaev et al. 2016, De Boer and Bernstein 2017, Dai et al. 2013). Moreover, a significant fraction of this literature relies on availability of accurate priors on some of the key process or workforce parameters (Abassi and Boukhris 2017, Tao et al. 2018, Heer and Bostock 2010, Dalvi et al. 2013, Laureti et al. 2006, Jung and Lease 2011, Rutchick et al. 2020), which are costly to obtain (De Boer 2017, Bonald and Combes 2016).

Within the larger domain of literature concerned with the quality of the outcomes and process costs in crowdsourcing, a small but notable subspace of work offers theoretical guarantees. The work of Berend and Kontorovich (2014) presents accuracy bounds for the weighted majority voting scheme. Khetan and Oh (2016) formulate a theoretical bound that describes the trade-off between budget and accuracy in voting processes with *adaptive task assignment* (i.e., tasks are assigned based on the data collected up until the moment of assignment, Barowy et al. (2012)). The authors introduce an adaptive assignment scheme that attains the formulated fundamental limit. Similarly, Manino et al. (2018) consider adaptive assignment of the next worker to an item. The authors' theoretical formulation of the accuracy gap between the uniform allocation of workers (Karger et al. 2014), an adaptive one, and an assignment that maximizes information gain (Simpson and Roberts 2015). They derive tight yet not exact bounds on the accuracy of these assignment policies. Livshits and Mytkowicz (2014) is an example of a theoretical work that considers costs of voting completion: the authors use power analysis to obtain ex-ante estimates for the number of votes required to resolve each item with a certain level of statistical significance.

In contrast with these papers, our work focuses on the  $\delta$ -margin consensus rule and provides an exact expression of probability of error, expected time until consensus as well as its variance, enabling a more nuanced consideration of the requesters' utility – one that considers worker *wages*.

While the latter is seldom addressed in the literature, a notable counterexample is the work of Singer and Mittal (2013). The authors present incentives-compatible mechanisms for maximizing the number of tasks subject to budget constraints, as well as for minimizing worker payments given a fixed number of tasks.

## 3.2. Literature on $\delta$ -margin voting

Literature on group decision-making and voting mechanisms recognizes two types of majority consensus vote: *simple* majority and *absolute* majority. The former is attained when the number of votes for option A is greater than the number of votes for the alternative option B ( $s_A > s_B$ ), while the latter requires that the number of votes for option A is greater than half of all votes ( $s_A > \frac{n}{2}$ ) (Laruelle and Valenciano 2011).<sup>3</sup> In the case of binary (or *dichotomic*) voting, the distinction between simple and absolute majorities only exists when some voters abstain or vote neutrally.

While the  $\delta$ -margin voting aggregation method is a salient choice among practitioners, crowdsourcing literature typically involves  $\delta$ -margin voting only in empirical or experimental settings, missing a chance for ex-ante theoretical evaluation of its benefits.<sup>4</sup>

In experimental comparisons of prototypical crowdsourcing tasks, De Boer (2017) remarks that while the  $\delta$ -margin method (called 'Beat-By-K' following Goschin (2014)) yields very accurate results for relatively high values of  $\delta$ , it is expensive to run in settings that prioritize utility without any budget constraints.

Several consensus aggregation methods in the literature capture a different but similar idea to  $\delta$ -margin voting – namely, requiring a stronger, more confident agreement among the workers. For example, in the 'Automan' scheme (Barowy et al. 2012) the requester continues sampling votes until a given statistical confidence value is attained by the voting process.

Dietrich and List (2007) generalize majority voting to a larger family of 'quota rules' (also known as 'k-unanimity' or 'k-majority' in cybernetics and discrete mathematics (Alon et al. 2006, Scheidler et al. 2015)), in which an item is assigned a certain label if and only if the number of workers voting for that label exceeds a given threshold k. In a situation of simultaneous vote by all workers within a fixed size workforce, the quota scheme becomes similar to  $\delta$ -margin voting, with two major differences: (a) there is no guarantee that at the end of the voting the margin will be satisfied, and (b) the process may accumulate more votes than necessary, a disadvantage when every vote has an associated cost (as in the case of paid crowdsourcing).

# 4. Theoretical characteristics of $\delta$ -margin voting processes

In this section, we provide closed-form theoretical formulations of key characteristics of the  $\delta$ -margin voting process. In particular, we employ well-known results from theory of Markov Chains to provide:

- a characterization of *resulting label quality* for  $\delta$ -margin majority crowdsourcing processes;
- the *expected number of votes* to reach consensus is attained (i.e., *expected time* until consensus);
- the variance of number votes to reach consensus;
- an expression for the *pdf* of time until consensus.

<sup>3</sup> Alternatively, the strength of majority condition is defined as a multiplier k on the required inequality requirements that are modified to  $s_A > k \cdot s_B$  for k > 1, and  $s_A > k \cdot n$  for  $\frac{1}{2} \le k \le 1$ .

<sup>&</sup>lt;sup>4</sup> Symptomatically, even outside of the crowdsourcing literature field the more notable papers that overview some of the theoretical properties of this consensus scheme refer to  $\delta$ -margin majority voting as 'the forgotten decision rule' (Llamazares 2006, García-Lapresta and Llamazares 2001). Some of the earliest – albeit brief – acknowledgements of the  $\delta$ -margin voting scheme can be found in Fishburn (2015) and Saari (1990).

## 4.1. Quality of the consensus vote

The first quantity of interest is the quality of the final consensus votes. The process always starts at state 0 (no votes). Let's denote as  $\mathbb{1}_C$  a random variable indicator of whether the consensus vote (in the event that it was reached) corresponds to the ground truth correct label of an item. We want to calculate the probability  $Q := P(\mathbb{1}_C = 1)$  of reaching terminal state  $\delta$  (i.e., voters reach consensus, and the correct label is assigned) vs. reaching terminal state  $-\delta$  (voters reach consensus, and the incorrect label is assigned, i.e.  $\mathbb{1}_C = 0$ ). In all other states, the voting continues.

THEOREM 1. For a  $\delta$ -margin voting process, with worker accuracy p and a consensus threshold  $\delta$ , the probability  $Q(\varphi, \delta)$  that the consensus vote c is correct, is:

$$Q(\varphi, \delta) := P(\mathbb{1}_C = 1) = \frac{\varphi^{\delta}}{1 + \varphi^{\delta}},\tag{1}$$

where  $\varphi = \frac{p}{1-p}$  are the *odds* of a single worker classifying the item correctly.

The proof is readily available in many introductory texts on Markov Chains (e.g., in (Feller 1968, page 344)) and we also provide it below, for completeness, and to also introduce notation that we use later in this paper.

**Proof.** Consider an asymmetric Gambler's Ruin model, where a gambler begins with the amount k of money and proceeds to gamble until she is either bankrupt, or wins the amount N > k. Let us formalize this model in the following way. Consider a series of random variables  $X_t, t \in \mathbb{N}_0, X_t = k + \xi_1 + \ldots + \xi_t$ , where

$$\xi_t = \begin{cases} 1 & \text{with probability } p. \\ 0 & \text{with probability } 1-p \end{cases}$$

We shall denote the *time of termination* of the game of a gambler who began with k amount of money (or k vote difference) as  $T_k := \inf t, t \ge 0 : X_t \in \{0, N\}$ . Consider the following identity:

$$\mathbb{E}[S^{X_t}|X_{t-1}, X_{t-2}, ..., X_1] = pS^{X_{t-1}+1} + (1-p)S^{X_{t-1}-1}.$$

A *martingale* is a process such that:

$$\mathbb{E}[Z_t | Z_{t-1}, ..., Z_1] = Z_{t-1}.$$

Let us choose S such that  $\{S^{X_t}\}_{t\geq 0}$  is a martingale. Namely, take  $S = \frac{1-p}{p}$  (note that the solution S = 1 is trivial). Using the fact that any martingale satisfies  $\mathbb{E}[Z_t] = \mathbb{E}[Z_0]$ , we obtain the equality

$$\mathbb{E}[S^{X_t}] = \mathbb{E}[S^{X_0}] = S^k.$$

In particular,  $\mathbb{E}[S^{X_{T_k}}] = S^k$ . At the same time,



Figure 2 Theoretical values of quality Q of consensus vote (Theorem 1) as a function of the probability of a correct answer p, for a fixed consensus threshold  $\delta$ .

$$\mathbb{E}[S^{X_{T_k}}] = S^N P(X_{T_k} = N) + S^0(1 - P(X_{T_k} = N)) = S^N P(X_{T_k} = N) + (1 - P(X_{T_k} = N))$$
(2)

Therefore,

$$P(X_{T_k} = N) = \frac{S^k - 1}{S^N - 1} \quad \text{when } p \neq 1/2.$$
(3)

In the context of a symmetric voting process, the initial value k is set equidistantly apart from the ruin value of zero and from the winning value N. The number of votes needed for either a consensus for the incorrect answer (ruin) or a consensus for the correct answer (winning) is  $\delta$ . Thus, substituting  $\delta$  for k and  $2\delta$  for N while recalling our choice of  $S = \frac{1-p}{p} = \frac{1}{\varphi}$ , we obtain the statement of the theorem.  $\Box$ 

Figure 2 plots the dependence of consensus vote quality Q on worker accuracy p for  $\delta \in \{2, 3, 4, 5\}$ .

**Discussion:** Note that the parameter  $\delta$  plays a significant role in the quality of the consensus vote: Based on Equation 1, the *odds* of the *consensus* vote being correct, are  $\varphi^{\delta}$ . Therefore by increasing  $\delta$ , we increase exponentially the odds that the consensus vote is correct. For example, consider a pool of workers with accuracy p = 0.75 (i.e.,  $\varphi = \frac{0.75}{1-0.75} = 3$ ). If we set  $\delta = 2$ , the expected quality of the overall classification will be Q(3, 2) = 0.9 (i.e., odds 9 to 1 being correct). If we increase  $\delta$  to  $\delta = 3$ , Q(3,3) = 0.964 (ie., odds 27 to 1), and if we increase to  $\delta = 4$ , then Q(3,4) = 0.9878 (i.e., odds 81 to 1). Next, we discuss how the cost of the process changes when we change  $\delta$ , and we show that we achieve exponential improvements in quality with a mostly linear increase in cost.

#### 4.2. Expected time until consensus

The next question is whether the process is guaranteed to terminate, and how many votes do we expect to collect until reaching a consensus (in other words, what is the *expected time* until consensus is reached). We can estimate the number of votes it takes to reach state  $\delta$  or state  $-\delta$  in terms of transitions in the Markov chain as follows. The number of remaining steps  $E_0$  from state 0 is one step it takes to reach the next state (i.e., state 1 with probability p or state -1 with probability 1-p) plus the number of remaining steps from this resulting node. By solving, we get:

THEOREM 2. The expected number of votes  $n_{votes}$  it takes to reach a (correct or incorrect) consensus when classifying an item using a  $\delta$ -margin voting scheme with worker accuracy p and consensus threshold  $\delta$  is:

$$\mathbb{E}[n_{votes}|\varphi,\delta] = \delta \cdot \frac{\varphi+1}{\varphi-1} \cdot \frac{\varphi^{\delta}-1}{\varphi^{\delta}+1},\tag{4}$$

where  $\varphi = \frac{p}{1-p}$  is the *odds* of a single worker classifying the item correctly.

**Proof.** Note that the expected number of votes needed to reach consensus when beginning at state k of a Markov chain is nothing but  $\mathbb{E}[T_k]$  in our earlier notation. Consider the equation

$$\mathbb{E}[X_{T_k}] = P(X_{T_k} = N) \cdot N + (1 - P(X_{T_k} = N)) \cdot 0 = N \cdot P(X_{T_k} = N).$$
(5)

Using Wald's equation for the expectation of a sum of random number of random variables, we obtain:

$$\mathbb{E}[X_{T_k}] = k + \mathbb{E}\left[\sum_{i=1}^{T_k} \xi_i\right] = k + \mathbb{E}[T_k]\mathbb{E}[\xi_i] = k + (\mathbb{E}[T_k])(p - (1-p)).$$
(6)

Combining (5) with (6) we get:

$$\mathbb{E}[T_k] = \frac{N}{2p-1}P(X_{T_k} = N) - \frac{k}{2p-1}$$

The statement of the theorem follows from making use of Equation 3 and the identity  $2p - 1 = \frac{\varphi - 1}{\varphi + 1}$  as well as performing the familiar substitutions:  $N = 2\delta$  and  $k = \delta$ .  $\Box$ 

**Discussion:** Note that the expected cost increases mostly linear with  $\delta$ . When  $\varphi$  gets close to 1 (i.e., mostly random workers), then the expected time to termination peaks at  $\delta^2$ . Figure 3 illustrates the dependence (4) of time until voting termination  $\mathbb{E}(n_{votes})$  on worker accuracy p, for a given consensus threshold  $\delta$ .

# 4.3. Variance of time until consensus

While the expected number of votes to completion is a useful metric for characterizing the voting process, it describes just the average across runs. In addition to the expectation, in reality we also want to know the robustness of a process and how reliably it will finish within the expected time frame. For this reason, it is important to know the *variance* of the process in terms of number of votes required to completion and, more generally, the overall pdf of the distribution. To derive these values, we rely on results from the theory of Markov Chains with absorbing states.



**Figure 3** Expected time to voting termination as a function of probability of correct answer p, for a fixed consensus threshold  $\delta$ .

We showed that a  $\delta$ -margin voting process can be modeled as a Markov random walk. In particular, a  $\delta$ -margin voting process with a consensus threshold  $\delta$  corresponds to a Markov chain process with two absorbing states and  $2(\delta - 1) + 1$  transient states. If one enumerates the states such that the two absorbing states come last, then the transition matrix of this process has the canonical form (Grinstead and Snell 1997):

$$\mathbf{M} = \begin{pmatrix} \mathbf{T}_{tr} & \mathbf{T}_A \\ \mathbf{0} & I_2 \end{pmatrix}$$

Here  $\mathbf{T}_{tr}$  is a matrix of size  $(2\delta - 1) \times (2\delta - 1)$  containing the probabilities of transitioning from a transient state *i* into a transient state *j*:

$$\mathbf{T}_{tr} = \begin{pmatrix} 0 \ p \ 0 \ 0 \ \cdots \ 0 \\ q \ 0 \ p \ 0 \ \cdots \ 0 \\ 0 \ q \ 0 \ p \ \cdots \ 0 \\ 0 \ 0 \ q \ 0 \ \cdots \ 0 \\ \vdots \ \vdots \ \vdots \ \ddots \ \vdots \\ 0 \ 0 \ 0 \ 0 \ q \ 0 \end{pmatrix};$$

 $\mathbf{T}_A$  is a  $(2\delta - 1) \times 2$  matrix of transition probabilities from transient into absorbing states:

$$\mathbf{T}_{A} = \begin{pmatrix} q \ 0 \ 0 \ \cdots \ 0 \ 0 \\ 0 \ 0 \ \cdots \ 0 \ p \end{pmatrix}^{\top};$$

 $I_2$  is a 2×2 identity matrix, and **0** is a 2×((2 $\delta$ -1)+1) null matrix. Using the above, we define the *fundamental matrix* of this absorbing matrix chain as:

$$\mathbf{N} := \sum_{k=0}^{\infty} \left( \mathbf{T}_{tr} \right)^k$$

It can be proven (Grinstead and Snell 1997) that:

$$\mathbf{N} = \left(\mathbf{I}_{2\delta-1} - \mathbf{T}_{tr}\right)^{-1}$$

The elements of **N** correspond to the expected number of visits to the state j, given that the initial state was i. The expected number of steps before consensus, when starting in transient state i, is the i-th element of the vector

$$\mathbf{t}=\mathbf{N}\cdot\mathbf{1}$$

where **1** is vector of ones of length  $2\delta - 1$ . Finally, we have:

PROPOSITION 1. The variance on the number of steps required to reach consensus when starting in a state i is the *i*-th entry of vector

$$\bar{v} = (2\mathbf{N} - \mathbf{I}_t)\mathbf{t} - \mathbf{t}_{sq},$$

where  $\mathbf{t}_{sq}$  is the vector of squared elements of  $\mathbf{t}$ .

Applying this known result to our model for  $\delta$ -margin voting, we offer a closed-form expression for the variance of time until consensus:

THEOREM 3. Variance of the number of votes it takes to reach consensus using  $\delta$ -margin voting process is:

$$Var(n_{votes}) = 4\delta\varphi \left(\frac{\varphi+1}{\varphi^{\delta}+1}\right)^2 \left[h(\delta) \cdot \varphi^{\delta-2} + \sum_{i=1}^{\delta-2} \left(h(\delta-i)(\varphi^{\delta+i-2}+\varphi^{\delta-i-2})\right)\right],\tag{7}$$

where coefficients h are defined using the floor function:

$$h(z) := \left\lfloor \frac{z^2}{4} \right\rfloor.$$

Stated differently,

$$h(z) = \begin{cases} \frac{z^2}{4}, & \text{if } z \text{ is even} \\ \frac{z-1}{2} \cdot \frac{z+1}{2}, & \text{if } z \text{ is odd.} \end{cases}$$

The sequence of coefficients defined by h(z) is also known as the *quarter squares* sequence and can be defined as the interleaving of square numbers and pronic numbers (Losanitsch 1897, Sloane 2019) (the latter defined as the product of two consecutive integers).

Equation 7 offers a simplification of the formula proposed by Anděl and Hudecová (2012), while agreeing with it numerically. Moreover, once the polynomial in square brackets is formulated for a given value of  $\delta$ , one can observe a simple and intuitive 'pyramidal' pattern that governs coefficient generation.

In order to provide the reader with intuition on the look of the polynomial in Equation 7, we list explicit formulas for  $Var(n_{votes})$  for the first few values of  $\delta$  in Table 1.





Figure 4 Left:  $Var(n_{votes})$  (left) and  $Var(n_{votes})/\mathbb{E}(n_{votes})$  (right) as a function of probability of correct answer p, for a fixed consensus threshold  $\delta$  (d).

Formulation in Equation 7 allows us to elaborate upon the plots in Figure 3 by adding bands of the size  $2\sqrt{Var(n_{votes})}$  to each trajectory of  $\mathbb{E}[n_{votes}]$ . Detailed plots for selected values of  $\delta$  are shown in Figure 5.

#### 4.4. Distribution of the time to consensus

Using the matrix notation used in the previous subsection, as well as known results about discrete phase-type distributions from queuing theory Neuts (1994), Latouche and Ramaswami (1999), we can fully describe the probability density function f(k) of time until consensus in  $\delta$ -margin voting.

REMARK 1. The event that a  $\delta$ -margin majority voting process will terminate after casting exactly *m* votes when starting from zero votes is characterized by the following probability



**Figure 5** Expected time until reaching consensus, with standard deviation bounds, for selected values of  $\delta$ .



Figure 6 Probability density of reaching consensus in exactly m steps (i.e., any consensus, correct or incorrect)  $f(T_{\delta} = m)$  for  $\delta = 4$ , as a function of the number of votes m, for varying worker odds  $\varphi$ .

density function:

$$f(T_{\delta} = m) = \mathbf{z} \cdot (\mathbf{T}_{tr})^{m-1} \cdot (\mathbf{T}_{A})$$
(8)

where  $\mathbf{z}$  is a vector of length  $2\delta - 1$  that encodes the initial state of the process. Since we mainly care for the case where we start with no votes, we set  $\mathbf{z} := (z_1, z_2, ..., z_{2\delta-1})$  with  $z_{\delta} = 1$  and  $z_i = 0$  for  $i \neq \delta$ .

The expression specifies separately the probability densities of the voting process terminating in each one of the two absorbing states – i.e., reaching a consensus with a correct label, as well as with an incorrect one. Using our derived expression for the probability of label correctness (Equation 1), one may sum the values in the matrix expression of  $f(T_{\delta} = m)$  in Equation 8, weighted by the respective probabilities of the consensus vote being correct/incorrect. Figure 6 visualizes a result of this computation for a fixed selected threshold level  $\delta = 4$  and for various values of  $\varphi$ .

# 5. Equivalence conditions for two $\delta$ -margin voting processes

In this section, we build in the results presented above, to explore the following questions:

• If we have two sets of workers, one with accuracy  $p_1$  and the other with accuracy  $p_2$ , how should we configure the voting process to generate results of similar aggregate quality? (Section 5.1)

• If we have two sets of workers, one with accuracy  $p_1$  and the other with accuracy  $p_2$ , how should we vary the worker payment, so that we get the same quality while keeping the overall cost the same? (Section 5.2)

• If workers can vary their effort, and therefore the time invested and the resulting quality, what type of effort-quality curves result in unique payment-maximizing points, and what type of effort-quality curves make workers indifferent to the quality-sensitive payment scheme? (Section 5.3)

#### 5.1. Quality equivalence for processes with different values of $\delta$

Suppose that we have two sets of workers, one with accuracy  $p_1$  and another with accuracy  $p_2$ . The first set of workers operate a  $\delta$ -margin majority voting scheme with threshold parameter  $\delta_1$ . Assuming, for now, that the employer only cares about the quality of the resulting work, what is the value of  $\delta_2$ , so that the set of workers with accuracy  $p_2$  generate the same quality of the results?

COROLLARY 1. If an item is classified by workers with accuracy  $p_1$  and threshold  $\delta_1$ , we can achieve the same quality of results by a set of workers with accuracy  $p_2$  if we set the threshold  $\delta_2$  to be:

$$\delta_2 = \delta_1 \cdot \frac{\ln \varphi_1}{\ln \varphi_2},\tag{9}$$

where  $\varphi_1 = \frac{p_1}{1-p_1}$  and  $\varphi_2 = \frac{p_2}{1-p_2}$  are the *odds* of a single worker classifying the item correctly. The result is obtained by setting  $Q(\varphi_1, \delta_1) = Q(\varphi_2, \delta_2)$  (from Equation 1) and solving for  $\delta_2$ .

#### 5.2. Worker Pay Equivalence

Assume that workers with accuracy  $\varphi$  are paid  $pay(\varphi)$  per vote. Then a reasonable notion of cost associated with labeling a single example is simply:

$$Cost(\varphi, \delta) = pay(\varphi) \cdot n_{votes}(\varphi, \delta),$$

where  $pay(\varphi)$  is constrained to take only non-negative values. Then, using Equation 2, the expected cost of classifying an item is:

$$\mathbb{E}[Cost|\varphi,\delta] = pay(\varphi) \cdot \delta \cdot \frac{\varphi+1}{\varphi-1} \cdot \frac{\varphi^{\delta}-1}{\varphi^{\delta}+1}$$
(10)

For now, we assume that an employer is *risk-neutral* and cares only about the expected quality of the result and the expected  $cost.^5$  A risk-neutral employer would like to pay teams of workers

 $<sup>^{5}</sup>$  It is reasonably easy to extend the results to the case of risk-averse requesters, by using our results for the variance of time until consensus (which is a proxy for the cost of the process).

with different accuracies  $p_1$  and  $p_2$  in a way that is fair: as long as the teams can generate results of equal utility to the requester, they should be paid the same total amount.

From Section 5.1, we can set  $\delta_2$  to adjust for the different worker accuracy  $p_2$ , thereby assuring that the quality of results is the same:  $Q(\varphi_1, \delta_1) = Q(\varphi_2, \delta_2)$ . Of course, as shown in Section 4.2, a different consensus threshold  $\delta_2$  also changes the expected number of votes required to reach consensus. When  $\delta_2 = \delta_1 \cdot \frac{\ln \varphi_1}{\ln \varphi_2}$ , we have results of equal quality, we can ensure equal costs by setting:

$$\mathbb{E}[Cost|\varphi_{1},\delta_{1}] = \mathbb{E}[Cost|\varphi_{2},\delta_{2}]$$
$$\mathbb{E}[Cost|\varphi_{1},\delta_{1}] = \mathbb{E}[Cost|\varphi_{2},\delta_{1}\cdot\frac{\ln\varphi_{1}}{\ln\varphi_{2}}]$$
$$pay(\varphi_{1})\cdot\delta_{1}\cdot\frac{\varphi_{1}+1}{\varphi_{1}-1}\cdot\frac{\varphi_{1}^{\delta_{1}}-1}{\varphi_{1}^{\delta_{1}}+1} = pay(\varphi_{2})\cdot\delta_{1}\cdot\frac{\ln\varphi_{1}}{\ln\varphi_{2}}\cdot\frac{\varphi_{2}+1}{\varphi_{2}-1}\cdot\frac{\varphi_{2}^{\delta_{1}\cdot\frac{\ln\varphi_{1}}{\ln\varphi_{2}}}-1}{\varphi_{2}^{\delta_{1}\cdot\frac{\ln\varphi_{1}}{\ln\varphi_{2}}}+1}$$
(11)

Given that  $\varphi_2^{\delta_1 \cdot \frac{\ln \varphi_1}{\ln \varphi_2}} = \varphi_1^{\delta_1}$ , we get the following corollary:

COROLLARY 2. If workers with response accuracy  $\varphi_1$  are paid  $pay(\varphi_1)$  per vote, then workers with accuracy  $\varphi_2$  will generate results of the same quality and cost if the ratio of the payments is:

$$\frac{pay(\varphi_1)}{pay(\varphi_2)} = \frac{\ln \varphi_1}{\ln \varphi_2} \cdot \frac{\varphi_2 + 1}{\varphi_1 + 1} \cdot \frac{\varphi_1 - 1}{\varphi_2 - 1},\tag{12}$$

where  $\varphi_i = \frac{p_i}{1-p_i}$  are the odds that a worker belonging to pool *i* classifies an item correctly.

# 5.3. Iso-payment effort function

Using the results of corollary 3, we can further investigate the relationship between payment per item and the per-item *effort* made by a worker. Consider that for a given worker, an unknown function  $p = g(\psi)$  governs the relationship between the effort  $\psi$  expended by the worker on an item and the accuracy p of the response given. (One possible measurable proxy for the level of effort is the amount of time spent working on an item.) Given our understanding of the relative amounts of payment, given in (12), we can connect payment to effort for a given function  $g(\psi)$ .

For example, assume that the relationship between a worker's effort expended and accuracy of the answers is  $p = e^{\psi} - 1$ . This particular accuracy-effort function is represented with the x-marked line on the left panel of Figure 7. Since we are interested in relative adjustments to payment rate, set  $pay(\varphi_2)$  in equation (12) to be equal to 1. We then get  $pay(\varphi) = c_0 \cdot ln(\varphi) \cdot \frac{\varphi-1}{\varphi+1}$ , where  $c_0$  is a constant, and  $\varphi$  is equal to  $\frac{p}{1-p}$  as before. We can now connect worker effort to the level of payment per unit of effort  $pay(\varphi)/\psi$  (if we continue our example of effort measured as time spent per item,



**Figure 7** Properties of various accuracy-effort curves, plotted for a range of accuracy values  $0.51 \le p \le 0.98$ . Left: sample iso-payment accuracy curves, contrasted with two accuracy functions chosen without iso-payment considerations:  $p = e^{\psi} - 1$  and  $p \sim tanh(\psi)$ . Right: visual demonstration of the iso-payment property.

this last quantity would represent *hourly wage*). The x-marked line on the right panel of Figure 7 represents the relationship for our particular chosen function  $p = g(\psi) = e^{\psi} - 1$ .

We now ask: what relationship between a worker's effort and the accuracy of her responses would make her payment invariant to effort expended? In other words, we are looking for such  $g(\psi)$  that the 'hourly wage'  $pay(\varphi)/\psi$  is independent of  $\psi$ . It is immediate that all such functions can be defined by

$$\psi = c \cdot \ln\varphi \cdot \frac{\varphi - 1}{\varphi + 1} =: g_{iso}^{-1}(\varphi), \tag{13}$$

where c is a constant. Figure 7 illustrates a sample of several functions  $p = g_i(\psi)$ , different only in the chosen values of the constant factor c.

A more realistic hypothesis about a relationship between worker accuracy  $\varphi$  and her effort  $\psi$  would incorporate an assumption of diminishing returns. Figure 7 also includes a plot of an accuracy function  $p = \frac{1}{2} + \frac{2}{5} \tanh(\frac{\psi}{4})$ , which conforms to this assumption<sup>6</sup>. Since a worker's natural objective is to maximize pay, the best strategy for a worker with this kind of a relationship between effort and accuracy would be to choose the global maximum of their hourly wage curve, if the global maximum is unique, and otherwise choose a global maximum corresponding to the lowest time expenditure.

#### 5.4. Utility-equalizing payments

We offered a way to set the pay rate for workers in each pool relative to the other pool, subject to *risk-neutral utility constraints*. A risk-neutral approach would entail a utility function that would only involve the expected cost of getting a consensus vote. However, a more nuanced notion of

 $^{6}$  The shape of a hyperbolic tangent is convenient for describing the natural assumption on the diminishing returns to effort. The constants are chosen to scale tanh with accordance to our context.

utility should consider not only the expected quality and expected number of votes required to reach consensus, but also the variance of each. Thus, a *risk-averse* procedure for establishing an appropriate equivalence for payment for a single item is as follows:

$$Q_1(\varphi_1, \delta_1) = Q_2(\varphi_2, x) \implies \delta_2 := x$$

$$U_{i} := \mathbb{E}[Q(\varphi_{i}, \delta_{i})] - \operatorname{pay}_{i}(\varphi_{i})\mathbb{E}[n_{votes,i}|\varphi_{i}, \delta_{i}] - \lambda \cdot \left(\sqrt{Var(Q(\varphi_{i}, \delta_{i}))} + \operatorname{pay}_{i}(\varphi_{i})\sqrt{Var(n_{votes,i})}\right)$$

$$\begin{cases}
U_{1} = U_{2} \\
Q_{1} = Q_{2}
\end{cases} \iff \operatorname{pay}_{2} := \operatorname{pay}_{1} \frac{\mathbb{E}[n_{votes,1}|\varphi_{1}, \delta_{1}] + \lambda\sqrt{Var(n_{votes,1})}}{\mathbb{E}[n_{votes,2}|\varphi_{2}, \delta_{2}] + \lambda\sqrt{Var(n_{votes,2})}} \tag{14}$$

A remark on notation: strictly, Q is not a random variable but a first moment of  $\mathbb{1}_C$ . We use the notation Var(Q) to mean, more strictly,  $Var(\mathbb{1}_C)$ . Although computing the term  $Var(Q(\varphi_i, \delta_i))$  is not necessary for using the payment equivalence condition in Equation 14, it is worth mentioning that the variance of the indicator of reaching a correct consensus is the variance of a Bernoulli random variable:  $Var(\mathbb{1}_C) = \mathbb{E}[\mathbb{1}_C^2] - (\mathbb{E}[\mathbb{1}_C])^2 = P(\mathbb{1}_C = 1)(1 - P(\mathbb{1}_C = 1)) = Q(1 - Q).$ 

The plots in Figure 8 below visualize the following process: for every fixed value of utility parameter  $\lambda$  and every triplet  $(p_1, \delta_1, p_2)$  (where  $0.5 < p_1 < 1, 0.5 < p_2 < 1$ ), we compute  $\delta_2$  that guarantees  $Q_1 = Q_2$ . We then derive the ratio of payments  $pay_2/pay_1$  for a given triplet, and we visually organize the results by the values of corresponding ratios of expected times to completion,  $\mathbb{E}[n_{votes}(p_1, \delta_1)]/\mathbb{E}[n_{votes}(p_2, \delta_2)]$ . The plots display the results for selected values of the utility parameter:  $\lambda = 1, \lambda = 2$ , and  $\lambda = 10$ .

# 6. Experimental validation

The results provided above are theoretical and asymptotic. How good are they in approximating the results of a real crowdsourced voting process with constrained resources? Obviously, in reality, some of the assumptions of our model will be violated: workers will not have identical qualities and items will have different degrees of difficulty. As the old adage states, *"all models are wrong"*, hence we want to see if our model is actually useful in providing guidance on how to set up a crowdsourcing process to achieve certain levels of quality for certain levels of cost.

We compare the theoretical results with those obtained from experiments. In particular, we compare the theoretical values of quality, completion time, and variance thereof with two kinds of simulations:  $\delta$ -margin voting simulated as *random walk*; and  $\delta$ -margin voting simulated as random draws from real-life *crowdsourced data*.



Figure 8 Utility-equalizing ratio of payment rates  $pay(\varphi_2)/pay(\varphi_1)$  for workers on two voting processes (14), where  $\delta_2$  is derived in a way that satisfies the equal quality constraint. Top left:  $\lambda = 1$ . Top right:  $\lambda = 2$ . Bottom:  $\lambda = 10$ .

# 6.1. Dataset description

We simulate a voting process using a dataset of real votes collected on a major crowdsourcing platform. We used the Bluebirds dataset (Welinder et al. 2010), which contains binary labels for 108 unique images solicited from 39 workers, as well as ground truth labels for each item (with a 44:56 size ratio of the True and False classes). The histogram in Figure 9 shows the empirical distribution of individual accuracy among workers, computed as the frequency of labeling an item correctly. The mean worker accuracy stands at 0.636. The distribution of computed per-item rates of correct response suggests that a third of items are difficult (i.e., frequency of correct responses across all workers for an item is below 0.5).

In the rest of this section, we simulate the  $\delta$ -margin voting process using simulated random walk as well as real life crowdsourced labels. We do so in order to both examine the asymptotic properties of experimental process quality and time to completion, as well as (and chiefly) to compare the experimental values with the theoretical results from Section 4. Indeed, a key assumption underpinning the theoretical results presented in this paper is that individual worker accuracies are distributed uniformly, with each worker having accuracy p. In real life crowdsourced data –as in the **Bluebirds** dataset– response accuracy varies across workers as well as across items (some being more difficult to classify). This is a deviation from our theoretical model, and the main purpose of the experiments in this section is to examine the extent to which empirical quantities obtained under such violations disagree with theoretical quantities.







**Figure 10** Relative error  $err(p) := Q_{sim}^{-1} \cdot (Q_{sim} - Q_{th}))$  of the average (expected) consensus vote quality, as a function of the probability of a correct answer p, for a range of fixed values of consensus threshold  $\delta$ .

## 6.2. Theoretical quality vs. experimental quality of label aggregation

The first question of interest is how closely the theoretical quality in Equation 1 captures a real or realistic process of  $\delta$ -margin voting in a crowdsourcing labeling task.

We begin by comparing the theoretical value of quality in Equation 1 with the quality resulting from random walk simulations of  $\delta$ -margin voting with the same accuracy p. For each value of delta  $\delta \in [2, 3, ..., 7]$  and for each value of p, we simulate 100,000 random walk trials and record the average correctness of the consensus vote. Figure 10 shows the relative error between the computed average quality of the vote and the quality estimated from simulations,  $err(p) := Q_{sim}^{-1} \cdot (Q_{sim} - Q_{th}))$ .

Next, we transition to comparing the theoretical values to estimations with *observational data* experiments. As discussed, the **Bluebirds** dataset violates several of our modeling assumptions (i.e., worker accuracy levels as well as item difficulty are not uniform). For this reason, estimating

quantities of interest using random draws from this data is an informative benchmark for assessing real-life usefulness of presented theoretical formulations.

For a given  $\delta \in [1, 2, ..., 11]$ , the general procedure that we used for simulating  $\delta$ -margin voting is as follows. For a given item, we iteratively draw votes from labels given by a worker pool on this item, until the agreement threshold  $\delta$  is attained. We run r such simulations of the  $\delta$ -margin voting process for each item.

When examining quality and time to completion of experimental voting results in isolation, we performed sampling without replacement, and we set the number of voting process repetitions per item to r = 20. We additionally simulated a range of conditions on worker pool accuracy and item difficulty. In other words, we ran  $\delta$ -margin voting simulations as described above for a range of pre-filtered versions of the Bluebirds dataset – subsets that included only items of certain maximum levels of difficulty and workers with a certain minimum level of accuracy. In particular, we required that the worker accuracy be above  $\tau_w$  and the item accuracy be above  $\tau_w$  (where pairs  $(\tau_w, \tau_i)$  take all values on a lattice  $[0, 0.1, ..., 0.9]^2$ ). Thus, for each of the values of the consensus threshold  $\delta$  we run 20 experiments for each of the items that passed a given worker-item accuracy filter. Our experimental process is described by the pseudocode procedure in Appendix A (Algorithm 1).

We first examine some of the asymptotic properties of quality estimates from our crowdsourced data experiments. Figure 11 illustrates the resulting quality of the labels aggregated in our experiments, summarized for all settings and grouped by levels of the filtering on the worker accuracy. The reason that the processes do not attain the Q = 1 limit –even for the highest filtering thresholds of worker pool accuracy– is likely that the quality is aggregated across experiments with various settings of *item* accuracy, including settings where no minimum threshold for item 'ease' is set. Thus, a non-negligible proportion of experiments (those without any filter on item accuracy) included difficult items which get incorrect consensus votes even when solicited from top worker pools.

Next, we will benchmark the results from crowdsourced data experiments against the theoretical values obtained from Equation 1. To obtain the experimental values for the comparison, we performed sampling with replacement<sup>7</sup> and used the entire Bluebirds dataset. For a given  $\delta$ , we simulate  $\delta$ -margin voting and compare the resulting consensus labels with the ground truth. We record the average correctness (frequency of matching the ground truth) of  $\delta$ -margin consensus labels across all items. This process of procuring consensus votes and aggregating their correctness across all items is repeated 100 times for each value of  $\delta$ , collecting 100 statistical quantities, each of which

<sup>&</sup>lt;sup>7</sup> This prevents termination of the voting process due to exhausting the total set of available votes for high values of  $\delta$ . Repeated exposure to items is also a realistic scenario in some crowdsourced tasks, although our voting simulation necessitates that a worker always responds the same way to the same item.



Figure 11 Correctness of resulting consensus vote, averaged across all experiments and all settings for a fixed consensus threshold  $\delta$  and fixed minimum worker accuracy threshold  $\tau_w$ .

summarizes consensus quality for the full set of items. (We also record an average number of votes it took to reach consensus for the full set of items – we discuss the results in subsequent sections.)

For each value of  $\delta$ , we also compute the theoretical value of consensus vote quality from Equation 1. To do this, we take the worker pool accuracy level p to be equal to the average of accuracies among all workers recruited for the task (regardless of whether a vote was solicited from them). For example, when recruiting the entire set of 39 workers to the task of labeling all 108 items using  $\delta$ -margin voting with threshold  $\delta = 3$ , the empirically sourced pool-level odds of correct response in the Bluebirds dataset is  $\varphi \approx \frac{0.6356}{1-0.6356} \approx 1.744$ , and hence the theoretical quality of the voting process is  $Q_{th}(\varphi, \delta) \approx \frac{1.744^3}{1+1.744^3} \approx 0.841$ .

Finally, we compute the difference between the experimental and theoretical values of result quality. The simulation process is described in Algorithm 2 in Appendix A.

Figure 12 demonstrates the results for the total, unfiltered pool of workers and items. Theoretical quality makes for a reasonably small overestimation of the empirical results quality (with the exception of  $\delta = 2$ , where part of the experiments demonstrated better-than-theoretical quality). The trend demonstrates that the gap between the theoretical quality derived earlier and the quality of (quasi-)empirical results is widening as the consensus majority requirement becomes stronger, however the variance of the difference becomes smaller. Figure 12 also includes an illustration of the same quantities under an identical voting with replacement simulation, but with pre-set quality filters on the pool of workers.

# 6.3. Theoretical vs. experimental time until completion

As before, we begin by comparing the theoretical value of time until consensus in Equation 4 with the average time (i.e., number of votes) cast in 100,000 random walk simulations, for each pair  $(p, \delta)$ .



Figure 12 Comparison between theoretical and empirical values of overall task result quality, for various values of consensus threshold  $\delta$  and a fixed (empirically derived) worker pool accuracy p. Each data point is an average over 100 experiments. Top left: violin plot of distributions of the difference  $Q_{th} - Q_{exp}$ . Top right: side-by-side comparison and difference  $Q_{th} - Q_{exp}$ . Bottom row: combined violin and point difference plots, for pre-filtered datasets with worker quality  $\tau_w > 0.5$ ,  $\tau_w > 0.7$ , and  $\tau_w > 0.9$  respectively.

Figure 13 demonstrates that the relative error  $err_{ET}(p) := \mathbb{E}_{sim}^{-1}[n_{votes}] \cdot (\mathbb{E}_{sim}[n_{votes}] - \mathbb{E}_{th}[n_{votes}]))$ between the computed theoretical values and the quality estimated from random walk simulations is small.

Again, we follow up by a comparison between theoretical and *empirical* data (as opposed to theoretical vs. *simulated* data, above). We use the same procedure as described in the previous section to compute the empirical values of the number of votes it took to reach consensus. We estimate the empirical value of accuracy p the same way as above in order to compute the theoretical value of expected number of votes using Equation 4.

The results of this comparison are illustrated in Figure 14. As one would intuitively expect, increasing the value of  $\tau_w$  (a lower bound on the accuracy of recruited workers) pushes the number of votes required for consensus to be closer to the threshold  $\delta$ . In other words, any sample of workers is likely to agree, so a series of  $\delta$  votes is likely to be enough to reach a  $\delta$  majority. Also unsurprisingly, the adherence to the  $N_{votes} = \delta$  line is tighter for smaller levels of  $\delta$ , and the graph begins to curve upwards for larger values of  $\delta$ . Indeed, the violin plot on the right panel of Figure 14



Figure 13 Relative error  $err_{ET}(p) := \mathbb{E}_{sim}^{-1}[n_{votes}] \cdot (\mathbb{E}_{sim}[n_{votes}] - \mathbb{E}_{th}[n_{votes}]))$  of the average (expected) time until consensus, as a function of the probability of a correct answer p, for a range of fixed values of consensus threshold  $\delta$ .



**Figure 14** Number of votes it took to reach consensus. Left: averaged across all experiments and all  $\tau_i$  settings for a fixed consensus threshold  $\delta$  and fixed minimum worker accuracy threshold  $\tau_w$ . Right: 'violin plot' of  $n_{votes}$  distributions for a fixed  $\delta$  across all settings of  $\tau_w$  and  $\tau_i$ , on a logarithmic scale.

illuminates the fact (which we confirmed directly by examining individual items in our computation) that, for higher threshold values  $\delta$ , only a few items within a given setting may occasionally take an enormous number of votes to reach consensus, and such 'outliers' drive up the mean value of  $N_{votes}$  across all settings in the left panel for higher values of  $\delta$ . We confirmed this fact explicitly by examining the way time to completion was distributed across items in our computations.

# 6.4. Theoretical variance of time until completion vs. random walk simulations

First, we compare the variance of time to termination in a simulated random walk with two absorbing states with the theoretical value of variance obtained as in Proposition 3. To obtain the 'experimental' variance, the first step is to simulate n trials of a Markov random walk with given parameters p and  $\delta$  defining the initial value in a chain with absorbing states at 0 and  $2\delta$ . We used



Figure 15 Relative error  $\ell(p) := Var_{sim}^{-1}(n_{votes}) \cdot (Var_{sim}(n_{votes}) - Var_{th}(n_{votes}))$  of the variance values between the theoretical and the simulated random walk experiments, as a function of the probability of a correct answer p, for a fixed consensus threshold  $\delta$ .

n = 100,000 trials for each pair of values  $(p, \delta)$ , recording the variance of the number of votes until consensus for each pair.

We are interested in establishing for which values of p does the theoretical variance of time to termination start diverging from the value obtained via random walk simulations. For a range of values of the threshold  $\delta$ , the plot in Figure 15 shows the relative error in the estimation of variance of a simulated random walk versus the theoretical values in Formula 7, as a function of  $p \in [0.51, 0.93]$ :  $\ell(p) := Var_{sim}^{-1}(n_{votes}) \cdot (Var_{sim}(n_{votes}) - Var_{th}(n_{votes}))$ .

# 7. Discussion and future work

Delta-margin majority voting is a frequent method of choice when aggregating worker votes on crowdsourcing platforms. Adaptively requiring a strong enough majority —instead of requesting a specific number of votes for consensus—, this method is favored in practice for promising higher accuracy of results without casting too large a number of votes when it is not necessary to do so. Yet wherever the preference for this method emerged ad hoc, a theoretical study of this method remained lacking.

In this work, we propose a novel way to model the  $\delta$ -margin voting process as a Markov process. We use well-established probability theory and statistics results to put forth closed-form expressions that quantify the expected quality of the consensus vote. We also formulate the PDF, expectation, and variance of the number of votes one needs to cast in order to reach consensus. Furthermore, we formulate a series of useful equivalences. The first of these is an equation that defines how one ought to set the threshold parameter of a voting process in order to attain the same quality as that of a process with different (given) accuracy and consensus threshold. The second equivalence defines a risk-neutral utility-equalizing ratio of payments for two workflows which differ in worker response accuracies. In other words, it suggests hot to set the wage for the second process, relative to the wage for the first one, in such a way that the two processes attain the same utility, defined here as combining result quality and expected processing time (a proxy for process cost). We further refine this result to a case of risk-averse cost function – one that considers both the expectation and the variance of time until consensus.

We probed the reliability of our theoretical results in real-life scenarios. Indeed, one limitation of our results is in making the assumption of uniform response accuracy among workers – an assumption not likely to hold on crowdsourced platforms. However, comparing each of our theoretical quantifications with corresponding simulations (random walk as well as, separately, draws from real crowdsourced labels) suggested that our theoretical results are robust to violations of the worker accuracy uniformity assumption. This is an encouraging finding for practitioners seeking to design cost-optimal crowdsourced workflows: theoretical quantifications can be used to guide process design ex ante, without having to tune the parameters of the voting process after a series of costly trial-and-error runs.

While our experimental evaluations bode well for the applicability of the theoretical model as it stands (i.e., with the assumption of a uniform accuracy of all workers for all items), one future work direction would be to extend our model to formally handle worker and item heterogeneity. What are the equivalent results to Theorems 1, 2 and the variance results in 4.3 that hold for voting tasks where worker accuracies are not constant and equal to p, but follow a distribution with mean p?

A further challenge to applicability that stems from our assumption about worker accuracy p is that the accuracy parameter is seldom fully observed in practice. The question of interest to practitioners is that of an inverse problem: can our theoretical results be used to infer the accuracy distribution of a pool of workers, given limited or no access to ground truth labels? In future work, powerful statistical packages such as TensorFlow may be employed to solve an inferential version of the workflow problem with real-life crowdsourced task data and unknown worker accuracy parameters.

Another limitation of our work is its focus on binary classification. A natural direction of future work is to derive corresponding results that hold for non-binary classification tasks<sup>8</sup> A further extension of this work could attempt to provide theoretical quality and cost estimates for continuous (regression) tasks.

Finally, in a broader outlook on future work, we plan on extending the analysis herein onto other types of processes, with a focus on understanding the interplay between quality, time to

<sup>8</sup> This question appears to be easily addressed by representing multi-class tasks as binary ones by pooling all but the correct class labels together.

completion, and payment costs. Ultimately, our goal is to have a set of 'modular' analyses that allow practitioners to take as input the structure of a crowdsourcing task or process and provide an ex ante prediction of its behavior, without having to execute the process to see the results.

Despite the fact that the voting process modeled using Gambler's Ruin random walk is guaranteed to terminate, the number of votes necessary for reaching consensus on a given item may be prohibitively large. This is one of the reasons for developing heuristics and, eventually, theoretical results for *escalation workflows*. In this type of a crowdsourced labeling process, an item will either get resolved by a pool of workers recruited from a given pool, or remain unresolved after certain maximum amount of iterations (an *escalation criterion*), after which the requester will transfer the item to the next 'stage' where higher accuracy labourers will vote to conclude the item's consensus label<sup>9</sup>. In relation to workflows, it is interesting to establish how many stages will resolve most of the items, given worker accuracy at each stage. Further, how many workers do we expect to recruit to resolve all items? What is the most cost-efficient stopping criterion? In this direction of future work, as with the current work, we aim to enable practitioners to infer cost-optimal parameters of multistage workflows ex ante.

<sup>&</sup>lt;sup>9</sup> Another motivation for establishing escalation criteria is driven by cost efficiency: there may exist such combinations of accuracies  $(p_1, p_2 \text{ and wage rates } (pay_1, pay_2)$  that make it strictly more cost-efficient for a requester to escalate items sooner, regardless of the reasonable expectation of those items being resolved by the first, less accurate worker pool.

## Appendix A: Procedures for simulating $\delta$ -margin voting using observational data.

In this appendix, we present pseudo-code that describes various procedures we used to simulate  $\delta$ -margin voting for various analyses described in Section 6. In particular, we describe the procedures we used for:

• examining asymptotic properties of *consensus vote quality* as a function of accuracy p for various values of the threshold  $\delta$ 

• examining asymptotic properties of *time until consensus* as a function of accuracy p for various values of the threshold  $\delta$ 

• comparing the experimental consensus vote quality (i.e., the average correctness of the consensus votes resulting from our simulations that use observational data) with the theoretical value of quality in Equation 1.

• comparing the experimental time (i.e., number of votes it took) until consensus with the theoretical value of expected time until consensus in Equation 4.

Algorithm 1 The procedure simulating  $\delta$ -margin voting using various subsets of observational data, for computing experimental quality and time until consensus.

1:	$\label{eq:procedure} procedure \ QualityAndTime_VariousWorkerPools(data = Bluebirds)$	
2:	for $ au_w \in [0, 0.1,, 0.9]$ do	
3:	for $ au_i \in [0, 0.1,, 0.9]$ do	
4:	data_subset $\leftarrow filter(\text{data}, \tau_w, \tau_i)$	
5:	$\mathbf{for}\delta\in[0,1,,11]\mathbf{do}$	
6:	$\mathbf{for} \text{ item} \in \{\text{data\_subset}\} \mathbf{do}$	
7:	$\mathbf{for}i\in[0,,19]\mathbf{do}$	
8:	$\texttt{this\_trial\_stats} \leftarrow \textit{SimulateDMarginVoting}(\textit{item}, \delta, \textit{replace} = \texttt{False})$	
9:	$correctness\_and\_time \leftarrow get(this\_trial\_stats, ['is\_correct', 'N\_votes'])$	
10:	all_trials_single_item $\leftarrow append$ (correctness_and_time)	
11:	end for	
12:	all_trials_all_items $\leftarrow append(all_trials_single_item)$	
13:	end for	
14:	all_deltas_all_trials_all_items $\leftarrow append(all\_trials\_all\_items)$	
15:	end for	
16:	$all\_itemfilters\_all\_deltas\_all\_trials\_all\_items \leftarrow append(all\_deltas\_all\_trials\_all\_items)$	
17:	end for	
18:	$all\_filters\_correctness\_and\_time \leftarrow append(all\_itemfilters\_all\_deltas\_all\_trials\_all\_items)$	
19:	end for	
20:	$return(all_filters\_correctness\_and\_time)$	
21:	21: end procedure	

Algorithm 2 The procedure of simulating  $\delta$ -margin voting using observational data, for comparing theoretical and experimental values of quality and time to consensus.

1: <b>procedure</b> QUALITYEXPERIMENTSVSTHEORY(data = Bluebirds)	
2: $p = WorkerAccuracy(data)$	
3: <b>for</b> $\delta \in [0,, 11]$ <b>do</b>	
4: $quality\_theoretical = ComputeTheoreticalQuality(\delta, p)$	
5: <b>for</b> experiment_id $\in [0,, 99]$ <b>do</b>	
6: for item $\in$ Bluebirds do	
7: consensus_vote_stats $\leftarrow SimulateDMarginVoting(\delta, item, replace = True)$	
8: consensus_vote_correctness $\leftarrow get($ consensus_vote_stats, 'is_correct')	
9: $all\_items\_correctness \leftarrow append(consensus\_vote\_correctness)$	
10: end for	
11: $quality\_experimental\_singleTrial \leftarrow average(all\_items\_correctness)$	
12: $quality\_experimental\_multi \leftarrow append(quality\_experimental\_singleTrial)$	
13: end for	
$14: theoretical\_vs\_experimental \leftarrow combine (quality\_experimental\_multi, quality\_theoretical)$	
15: theory_vs_100Experiments_AllDeltas $\leftarrow append$ (theoretical_vs_experimental)	
16: end for	
17: $return$ (theory_vs_100Experiments_AllDeltas)	
18: end procedure	

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